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Multiple gravity–capillary wave forms near the minimum phase speed

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The existence of solitary waves near the minimum phase speed for waves in the gravity–capillary regime triggered our search for additional wave forms. We show that the governing Schrödinger-type equation also has a rich family of periodic solutions, and a preliminary study of these solutions is the objective of the present note.

1. Introduction

Longuet-Higgins (1989) was the first to provide numerical evidence that symmetric solitary waves are possible on a fluid of infinite depth, in the neighbourhood of the minimum phase speed of gravity–capillary waves. Ever since, these rather special waves have been the subject of intense study.

Vanden-Broeck & Dias (1992) computed two types of symmetric solitary waves near the minimum phase speed in infinite water depth, these being the ‘elevation’ and the ‘depression’ waves, and thus extended the work of Longuet-Higgins (1989), which focused on the ‘depression’ type only.

Later Akylas (1993) and Longuet-Higgins (1993) found that these solitary waves may be interpreted as particular envelope-soliton solutions of the corresponding Schrödinger equation. At the minimum phase speed it turns out that the phase speed equals the group velocity, and thus the envelope travels with the same speed as the carrier wave itself. The ‘elevation’ or ‘depression’ waves emerge when the maximum of the envelope coincides with the maximum or the minimum of the carrier wave, respectively.

Theoretical support for the asymptotic and numerical studies cited above was obtained by Iooss & Kirchgässner (1990) and Iooss & Kirrmann (1996), who provided rigorous existence proofs for small-amplitude symmetric solitary waves at finite and infinite water depth, respectively.

Asymmetric solitary waves, where the maximum of the envelope does not coincide with a maximum or a minimum of the carrier wave, require special treatment, as stated in several of the above cited studies. By using a revised perturbation technique, which takes into account exponentially small terms, Yang & Akylas (1997) showed that the solution had an extra term, which grows exponentially in space. This extra term, however, vanishes in the cases of symmetric solitary waves, justifying the results of e.g. Akylas, Dias & Grimshaw (1998).

In a related effort, Benjamin (1992) proposed an integro-differential equation for interfacial waves in a two-fluid system, taking the lower fluid to be of infinite depth and taking into account that the interface is subject to capillary effects. Recently, Benjamin (1996) discussed solitary waves of this evolution equation, and he furthermore provided an existence proof that this evolution equation also exhibited periodic solutions of a new type. Unfortunately, Benjamin did not set up an explicit expression for this periodic solution. However, Akylas *et al.* (1998) transformed Benjamin's evolution equation into a Schrödinger-type evolution equation, similar to the one studied in the present note.

Symmetric as well as asymmetric periodic wave forms in the gravity–capillary regime have been studied numerically by Zufiria (1987), and as described in Dias (1994) a periodic wave form bifurcates into the two types of symmetric solitary waves, when approaching the minimum phase speed.

In this study, we focus our attention on periodic wave forms near the minimum phase speed of gravity–capillary waves. The governing equations are formulated in § 2, the periodic solutions are derived in § 3 studied in § 4, and in § 5 the periodic solutions are compared to those computed by e.g. Zufiria (1987).

2. Governing equations

Linear gravity–capillary waves on the surface of a deep fluid obey the dispersion relation

$$\omega^2 = gk + sk^3,$$

where ω and k denote the angular frequency and the wavenumber of infinitesimal periodic waves, respectively, g is the gravitational acceleration, and s is the coefficient of surface tension σ divided by the fluid density ρ.

We follow Vanden-Broeck & Dias (1992) and Akylas *et al.* (1998), and use dimensionless variables based on s/c^2 as the characteristic length scale and s/c^3 as the characteristic time scale, c being the corrected phase speed. Thus, the dimensionless corrected phase speed is 1 identically.

The dimensionless dispersion relation takes the form

$$\omega^2 = |k|(\alpha + k^2), \tag{2.1}$$

where

$$\alpha = \frac{gs}{c^4} \tag{2.2}$$

is a dimensionless parameter.

Accounting for nonlinear and dispersive effects, the envelope of a weakly nonlinear gravity–capillary wave on a deep fluid is governed by the cubic nonlinear Schrödinger equation (NLS) which is correct to third order in the wave steepness ($\epsilon = ka$), a being the wave amplitude. See, for example Djordjevic & Redekopp (1977). An equation that includes effects up to fourth order was derived by Dysthe (1979) for pure gravity waves on a deep fluid. Hogan (1985) extended Dysthe's equation to gravity–capillary waves on a deep fluid, starting his derivation from Zakharov's (1968) integral equation.

In terms of the dimensionless variables used here, the equation derived by Hogan (1985) for the envelope $A(x,t)$ of the free-surface elevation η

$$\eta(x,t) = \tfrac{1}{2}\epsilon\left(A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + A^*\mathrm{e}^{-\mathrm{i}(kx-\omega t)}\right) + O(\epsilon^2) \tag{2.3}$$

takes the form

$$\mathrm{i}A_T + pA_{XX} + qA^2A^* + \mathrm{i}\epsilon(rA_{XXX} + uA^2A_X^* + v|A|^2A_X) - \epsilon kA\overline{\phi}_X|_{Z=0} = 0 \tag{2.4}$$

as given by Akylas *et al.* (1998). Here $X = \epsilon(x - c_g t)$, $Z = \epsilon z$, and $T = \epsilon^2 t$ are variables that describe the wave train's modulations in a frame of reference moving with the group velocity $c_g (= \partial\omega/\partial k)$; i is the imaginary unit, and the asterisk denotes complex conjugation.

To leading order in the wave steepness $\epsilon \ll 1$, equation (2.4) reduces to the familiar cubic NLS. The first term in (2.4), which is proportional to ϵ, includes higher-order modulation terms; whereas the last term in (2.4) proportional to ϵ reflects the interaction with the induced mean flow. The induced mean flow is described by the velocity potential $\overline{\phi}$, which satisfies the following boundary value problem:

$$\left.\begin{aligned} &\overline{\phi}_{XX} + \overline{\phi}_{ZZ} = 0 \quad (-\infty < Z \leqslant 0, -\infty < X < \infty),\\ &\overline{\phi}_Z = \tfrac{1}{2}\omega(|A|^2)_X \quad (Z = 0),\\ &|\nabla\overline{\phi}| \to 0 \quad (Z \to -\infty), \end{aligned}\right\} \tag{2.5}$$

∇ being the gradient operator ($\nabla = (\partial/\partial X, \partial/\partial Z)$). In solving this boundary value problem, we first apply a horizontal Fourier transformation to (2.5), which reduces the problem to a linear second-order differential equation with corresponding boundary conditions. By putting $Z = 0$ in the solution, applying the inverse Fourier transformation and differentiating with respect to X, we obtain

$$\overline{\phi}_X|_{Z=0} = -\frac{1}{2}\int_{-\infty}^{\infty} |k|\mathrm{e}^{\mathrm{i}kX}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kX}|A|^2\mathrm{d}X\right)\mathrm{d}k. \tag{2.6}$$

The coefficients of the rest of the terms in (2.4) are given by the following expressions:

$$p = \frac{\omega}{8k^2}\frac{3k^4 + 6\alpha k^2 - \alpha^2}{(\alpha + k^2)^2}, \tag{2.7a}$$

$$q = -\frac{\omega k^2}{16}\frac{2k^4 + \alpha k^2 + 8\alpha^2}{(\alpha - 2k^2)(\alpha + k^2)}, \tag{2.7b}$$

$$r = -\frac{\omega}{16k^3}\frac{(\alpha - k^2)(k^4 + 6\alpha k^2 + \alpha^2)}{(\alpha + k^2)^3}, \tag{2.7c}$$

$$u = \frac{\omega k}{32}\frac{(\alpha - k^2)(2k^4 + \alpha k^2 + 8\alpha^2)}{(\alpha - 2k^2)(\alpha + k^2)^2}, \tag{2.7d}$$

$$v = -\frac{3\omega k}{16}\frac{4k^8 + 4\alpha k^6 - 9\alpha^2k^4 + \alpha^3k^2 - 8\alpha^4}{(\alpha - 2k^2)^2(\alpha + k^2)^2}, \tag{2.7e}$$

where ω is determined from the dispersion relation (2.1).

3. Solutions

We are looking for solutions to (2.4) in the form

$$A = R(X)\exp\{\mathrm{i}(\lambda T + \epsilon f(X))\}, \tag{3.1}$$

where $R(X)$ and $f(X)$ are real functions of X and λ is a constant.

Substituting (3.1) and perturbation series for R, f and the mean flow potential $\overline{\phi}$

$$R = R_0 + \epsilon R_1 + \cdots, \quad f = f_0 + \cdots, \quad \overline{\phi} = \overline{\phi}_0 + \cdots$$

into (2.4), and separating the equation in real and imaginary terms and by order, we obtain a hierarchy of equations.

The lowest-order real terms yields

$$pR_{0,XX} - \lambda R_0 + qR_0^3 = 0, \tag{3.2}$$

and the lowest-order imaginary terms are satisfied identically.

At the next order, we find that the real and imaginary terms yield

$$pR_{1,XX} - \lambda R_1 + 3qR_0^2 R_1 = kR_0 \overline{\phi}_{0,X}|_{Z=0} \tag{3.3a}$$

and

$$pR_0 f_{0,XX} + 2pR_{0,X} f_{0,X} + rR_{0,XXX} + (u+v)R_0^2 R_{0,X} = 0, \tag{3.3b}$$

which determine R_1 from R_0 and $\overline{\phi}_{0,X}|_{Z=0}$ (which by substitution of (3.1) and the perturbation series into (2.6) can be determined from R_0), and f_0 from R_0. At the following orders $j = 2, 3, \ldots$, we obtain differential equations to determine R_j and f_{j-1} from lower-order quantities successively. In this short note we however focus on deriving expressions for the lowest-order contribution to R and f only.

Assuming that $pq > 0$, a solution to (3.2) is

$$R_0 = \mathrm{dn}\left\{ \left(\frac{q}{2p} \right)^{1/2} X, m \right\} \tag{3.4}$$

with

$$\lambda = \frac{2-m}{2} q. \tag{3.5}$$

Here $\mathrm{dn}\{U, m\}$ is a Jacobian elliptic function and m $(0 \leqslant m \leqslant 1)$ is a parameter which determines the form of the dn-function (or the envelope), see Abramovitz & Stegun (1972). Generally, the dn-function is $2\mathrm{K}(m)$ periodic, $\mathrm{K}(m)$ being the complete elliptic integral of the first kind. This solution has been obtained as a solution to the cubic Schrödinger equation in the gravity wave regime by e.g. Yuen & Lake (1982), but it has not been studied in detail in the gravity–capillary regime.

For $m = 0$ the dn-function simplifies significantly to $\mathrm{dn}\{U, m\} = 1$ with $\lambda = q$, and the solution then yields a Stokes-type wave. Half the period is found to be $\mathrm{K}(m) = \frac{1}{2}\pi$ even though it makes no sense to define a period for a constant function.

For $m = 1$ the dn-function simplifies to $\mathrm{dn}\{U, m\} = \mathrm{sech}\{U\}$ with $\lambda = \frac{1}{2}q$, and the solution then yields an envelope soliton. Half the period $\mathrm{K}(m)$ tends towards infinity, which is reasonable as there is no periodicity for an envelope soliton.

The solution (3.4) can be verified by insertion in (3.2), and as indicated above, the solution is in agreement with Akylas *et al.* (1998) for $m = 1$.

By inserting (3.4) into (3.3*b*), we obtain a linear and inhomogeneous second-order differential equation for f_0

$$\begin{aligned} p\,\mathrm{dn}\left\{ \left(\frac{q}{2p} \right)^{1/2} X, m \right\} f_{0,XX} &+ 2p \left(\mathrm{dn}\left\{ \left(\frac{q}{2p} \right)^{1/2} X, m \right\} \right)_X f_{0,X} \\ &+ r \left(\mathrm{dn}\left\{ \left(\frac{q}{2p} \right)^{1/2} X, m \right\} \right)_{XXX} \\ &+ (u+v)\mathrm{dn}^2 \left\{ \left(\frac{q}{2p} \right)^{1/2} X, m \right\} \left(\mathrm{dn}\left\{ \left(\frac{q}{2p} \right)^{1/2} X, m \right\} \right)_X = 0. \end{aligned} \tag{3.6}$$

The differential equation, which at first sight seems rather complex, is solved by using standard integration techniques.

For $0 < m < 1$, we first multiply (3.6) with $\mathrm{dn}\{U, m\}$ and by using straightforward integration techniques, we obtain

$$p\,\mathrm{dn}^2\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\} f_{0,X} + \frac{qr}{4p}(2-m)\,\mathrm{dn}^2\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\} + \frac{p(u+v)-3qr}{4p}\mathrm{dn}^4\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\} = C_1. \quad (3.7)$$

Then we divide (3.7) by $p\,\mathrm{dn}^2\{U, m\}$ and integrate once more, to obtain

$$f_0 = \frac{C_1}{p(1-m)}\left(\left(\frac{2p}{q}\right)^{1/2} \mathrm{Z}\left\{\left(\frac{q}{2p}\right)^{1/2} X + \mathrm{K}(m), m\right\} + \frac{\mathrm{E}(m)}{\mathrm{K}(m)}X\right) + C_2 - \frac{qr}{4p^2}(2-m)X - \frac{p(u+v)-3qr}{4p^2}\left(\left(\frac{2p}{q}\right)^{1/2} \mathrm{Z}\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\} + \frac{\mathrm{E}(m)}{\mathrm{K}(m)}X\right). \quad (3.8)$$

Here $\mathrm{Z}\{U, m\}$ is the Jacobian Zeta function, which is $2\mathrm{K}(m)$ periodic too. $\mathrm{E}(m)$ is the complete elliptic integral of the second kind, and C_1 and C_2 are arbitrary constants of integration.

For $m = 0$ the differential equation (3.3*b*) simplifies significantly to $pf_{0,XX} = 0$, and the solution is $f_0 = C_1 X + C_2$. This is in agreement with (3.8) for $m \to 0$, where $\mathrm{Z}\{U, m\} \to 0$, $\mathrm{K}(m) \to \frac{1}{2}\pi$ and $\mathrm{E}(m) \to \frac{1}{2}\pi$.

For $m = 1$ we use the same integration technique as for $0 < m < 1$, but by assuming that f_0 is bounded for $X \to \infty$, it is seen that the left-hand side of (3.7) tends towards zero and like Longuet-Higgins (1993), we put $C_1 = 0$ to accomplish that. As $Z\{U, m\} \to \tanh\{U\}$, $\mathrm{K}(m) \to \infty$ and $\mathrm{E}(m) \to 1$ for $m \to 1$, the solution for f_0 simplifies to

$$f_0 = C_2 - \frac{qr}{4p^2}X - \frac{p(u+v)-3qr}{4p^2}\left(\frac{2p}{q}\right)^{1/2} \tanh\left\{\left(\frac{q}{2p}\right)^{1/2} X\right\}, \quad (3.9)$$

which, except for the constant of integration, is in agreement with Akylas *et al.* (1998).

By using (2.3), (3.1), (3.4), and (3.5), the surface elevation correct to lowest order in ϵ can be written as

$$\eta = \epsilon\,\mathrm{dn}\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\} \cos\{kx - \Omega t + \epsilon f_0(X)\}, \quad (3.10)$$

where $f_0(X)$ has not been inserted in order to avoid too bulky expressions. Here

$$\Omega = \omega - \frac{2-m}{2}q\epsilon^2 \quad (3.11)$$

is the corrected angular frequency ($0 \leqslant \epsilon \ll 1$).

4. More details about the periodic solution

4.1. *The coefficients*

The group velocity of the waves is

$$c_g = \frac{\partial \omega}{\partial k} = \frac{\omega}{2k}\frac{\alpha + 3k^2}{\alpha + k^2}, \tag{4.1}$$

which for linear waves ($\epsilon = 0$) becomes equal to the phase velocity ($c = 1$) when $k = k_0 = \frac{1}{2}$, $\omega = \omega_0 = \frac{1}{2}$ and $\alpha = \alpha_0 = \frac{1}{4}$. This is seen by solving (2.1) and (4.1) for $\omega = k$ and $c_g = 1$. The rest of the coefficients then take the values $p_0 = \frac{1}{2}$, $q_0 = 11/(16)^2$ $r_0 = u_0 = 0$ and $v_0 = \frac{3}{32}$, from which it is found that $p_0 q_0 > 0$, making the solution (3.4) and (3.8) valid.

For small but finite waves ($0 < \epsilon \ll 1$) the corrected phase speed is given by

$$c = \frac{\Omega}{k} = \frac{\omega}{k} - \frac{2-m}{2}\frac{q}{k}\epsilon^2,$$

which for $c = 1$ yields that $\Omega = k$.

To estimate the values of k, α and thereby ω for which the corrected phase speed and the group velocity are the same, we expand the wavenumber k and the parameter α in series:

$$k = k_0 + \epsilon^2 k_1, \tag{4.2}$$

and

$$\alpha = \alpha_0 + \epsilon^2 \alpha_1. \tag{4.3}$$

These are inserted in Taylor expansions of (4.1) and (3.11):

$$c_g(k_0, \alpha_0) + \left.\frac{\partial c_g}{\partial k}\right|_{(k_0,\alpha_0)} \epsilon^2 k_1 + \left.\frac{\partial c_g}{\partial \alpha}\right|_{(k_0,\alpha_0)} \epsilon^2 \alpha_1 = 1, \tag{4.4}$$

$$\omega(k_0, \alpha_0) + \left.\frac{\partial \omega}{\partial k}\right|_{(k_0,\alpha_0)} \epsilon^2 k_1 + \left.\frac{\partial \omega}{\partial \alpha}\right|_{(k_0,\alpha_0)} \epsilon^2 \alpha_1 - \frac{2-m}{2} q\epsilon^2 = k_0 + \epsilon^2 k_1. \tag{4.5}$$

For both of the above equations the lowest-order terms are fulfilled by choosing k_0, α_0 and thereby ω_0 according to the results for $\epsilon = 0$. From (4.1) we find that $\partial c_g/\partial\alpha|_{(k_0,\alpha_0)} = 0$, the second-order terms in (4.4) yield $k_1 = 0$, and the second-order terms in (4.5) then yield

$$\left.\frac{\partial \omega}{\partial \alpha}\right|_{(k_0,\alpha_0)} \alpha_1 = \frac{2-m}{2} q.$$

By using (2.1) for $k_0 = \omega_0 = \frac{1}{2}$, we find that

$$\alpha_1 = (2-m)q. \tag{4.6}$$

With the new estimates for the wavenumber k and the parameter α it is possible to determine the new estimates for ω, p, q, r, u and v. Correct to ϵ^2 the angular frequency ω takes the simple form

$$\omega = \omega_0 + \epsilon^2 \omega_1,$$

where

$$\omega_1 = (2-m)q\omega_0,$$

as found by inserting (4.2), (4.3) and the values $k_0 = \frac{1}{2}$, $k_1 = 0$, $\alpha_0 = \frac{1}{4}$ and (4.6) in a Taylor expansion of the dispersion relation (2.1).

By inserting the calculated coefficients into (3.11) and (4.1), we find that $\Omega = \frac{1}{2} = k$ and $c_g = 1 = c$ (correct to second order), as it should be.

The sign of the product pq which is crucial for the validity of the solution (3.4) does not change within the region considered $0 \leqslant m \leqslant 1$ and $0 \leqslant \epsilon \ll 1$.

For given values of ϵ and m, we can calculate α by using (4.3) and (4.6). With g, s and α the physical value of the corrected phase velocity c can be calculated from (2.2). Now, physical values for the wavenumber k and the corrected frequency Ω can be calculated from the normalization, i.e. from $k = \frac{1}{2}c^2/s$ and $\Omega = \frac{1}{2}c^3/s$. Finally the wave amplitude a can be calculated from the definition of the wave steepness ϵ, i.e. $a = \epsilon/k$.

4.2. *Simplifications of the f-function*

It appears to us that it is not necessary to perform a revised perturbation analysis similar to that of Yang & Akylas (1997) including exponentially small terms for $0 \leqslant m < 1$. This is because the envelope minimum, even though it may be small, will be much greater than any exponential small term. However, we restrict ourselves to symmetric solutions.

In order to obtain periodicity and symmetry for the wave packet, we furthermore assume that the wave crest coincides with the maximum for the envelope, and that there are n waves (n integer) per modulation wavelength. Then we can determine the constants of integration C_1 and C_2 appearing in (3.8) and thereby f_0.

Starting from the expression for the surface elevation correct to lowest order in ϵ (3.8) and (3.10) it is first worth noticing that the x- and t-terms appearing in the argument of the cosine can, because $k = \Omega = \frac{1}{2}$ ($c = 1$), and $c_g = c = 1$, be combined into a single X-term, yielding

$$\eta = \epsilon \mathrm{dn}\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\} \cos\left\{\tfrac{1}{2}\epsilon^{-1}X + \epsilon f_0(X)\right\}.$$

For $0 < m < 1$, we for simplicity consider $X_1 = 0$ and $X_2 = \left((2p)/q\right)^{1/2} 2\mathrm{K}(m)$, corresponding to two succeeding envelope nodes, where the $\mathrm{dn}\{U, m\}$ simplifies to 1 and $\mathrm{Z}\{U, m\}$ simplifies to 0. By forcing the argument of the cosine to be 0 at X_1 and $2n\pi$ at X_2, where n is a positive integer, we can determine the constants of integration, C_1 and C_2 as

$$C_1 = (1-m)p\frac{\mathrm{K}(m)}{\mathrm{E}(m)}\left(\frac{(q/2p)^{1/2} n\pi}{\mathrm{K}(m)\epsilon} - \frac{k}{\epsilon^2} + \frac{qr}{4p^2}(2-m) + \frac{p(u+v)-3qr}{4p^2}\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right),$$

and

$$C_2 = 0.$$

By substituting these into (3.8), f_0 is found to be given by

$$\begin{aligned} f_0 = {} & \frac{\mathrm{K}(m)}{\mathrm{E}(m)}\left(\frac{(q/2p)^{1/2} n\pi}{\mathrm{K}(m)\epsilon} - \frac{k}{\epsilon^2} + \frac{qr}{4p^2}(2-m) + \frac{p(u+v)-3qr}{4p^2}\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right) \\ & \times \left(\frac{2p}{q}\right)^{1/2} \mathrm{Z}\left\{\left(\frac{q}{2p}\right)^{1/2} X + \mathrm{K}(m), m\right\} \\ & + \frac{\mathrm{K}(m)}{\mathrm{E}(m)}\left(\frac{(q/2p)^{1/2} n\pi}{\mathrm{K}(m)\epsilon} - \frac{k}{\epsilon^2}\right) X \\ & - \frac{p(u+v)-3qr}{4p^2}\left(\frac{2p}{q}\right)^{1/2} \mathrm{Z}\left\{\left(\frac{q}{2p}\right)^{1/2} X, m\right\}. \end{aligned} \qquad (4.7)$$

In order to keep $f_0(X)$ as an $O(1)$-function and prevent it from being an $O(\epsilon^{-2})$-function in X, we must further demand that

$$\frac{(q/2p)^{1/2}\,n\pi}{\mathrm{K}(m)\epsilon}-\frac{k}{\epsilon^2}=0, \tag{4.8}$$

which restricts the number of free parameters from the three (ϵ, m and n) to only two of these. This restriction will be illustrated in more details in the next section. The final result for $f_0(X)$ is

$$f_0=\left(\frac{qr}{4p^2}(2-m)\frac{\mathrm{K}(m)}{\mathrm{E}(m)}+\frac{p(u+v)-3qr}{4p^2}\right)\left(\frac{2p}{q}\right)^{1/2}\mathrm{Z}\left\{\left(\frac{q}{2p}\right)^{1/2}X+\mathrm{K}(m),m\right\}$$
$$-\frac{p(u+v)-3qr}{4p^2}\left(\frac{2p}{q}\right)^{1/2}\mathrm{Z}\left\{\left(\frac{q}{2p}\right)^{1/2}X,m\right\}. \tag{4.9}$$

For $m=0$ the solution is given by $f_0=C_1X+C_2$, and none of the constants can be determined by using the above arguments.

For $m=1$ the solution is given by (3.9). As asymmetric solitary waves require special treatment, cf. Yang & Akylas (1997): the constant of integration C_2 has to be put equal to zero in order to have the maximum of the envelope coinciding with a maximum of the carrier wave. Thus the solution is

$$f_0=-\frac{qr}{4p^2}X-\frac{p(u+v)-3qr}{4p^2}\left(\frac{2p}{q}\right)^{1/2}\tanh\left\{\left(\frac{q}{2p}\right)^{1/2}X\right\}, \tag{4.10}$$

which is in agreement with Akylas *et al.* (1998).

4.3. *The restriction on ϵ, m and n*

The restriction (4.8) means that only two of the three parameters (ϵ, m and n) can be chosen freely. The restriction (4.8) is mapped as curves of constant (integer) n in figure 1 with ϵ as the abcissa and m as the ordinate. As an upper limit for ϵ we have chosen $\epsilon=1$, which is far below the maximum steepness for a gravity–capillary wave ($\epsilon_{max}=2.3$ as suggested by Crapper 1984) and identical to the absolute upper limit due to the perturbation techniques used. The curves have been truncated near $m=0$ and near $m=1$ to indicate that the curves do not reach these values. This non-uniformity in the limits $m\to 0,\,1$ results from the fact that n is meaningless at these limits.

It is seen from the figure that on curves of constant n, m increases with the increasing of ϵ and that the curves tend asymptotically towards $m=1$ for ϵ 'large'. As a numerical example, which will be studied throughout the rest of this paper, we choose $\epsilon=0.5$ and $n=4$, which yield $m\approx 0.932$.

4.4. *The surface elevation*

As mentioned in §4.2 the expression for the surface elevation correct to lowest order in ϵ (3.8) and (3.10) simplifies further to

$$\eta=\epsilon\,\mathrm{dn}\left\{\left(\frac{q}{2p}\right)^{1/2}X,m\right\}\cos\left\{\tfrac{1}{2}\epsilon^{-1}X+\epsilon f_0(X)\right\} \tag{4.11}$$

when the group velocity and the phase velocity are identical ($c=c_g=1$). Here the coefficients q and p vary slightly with ϵ and m.

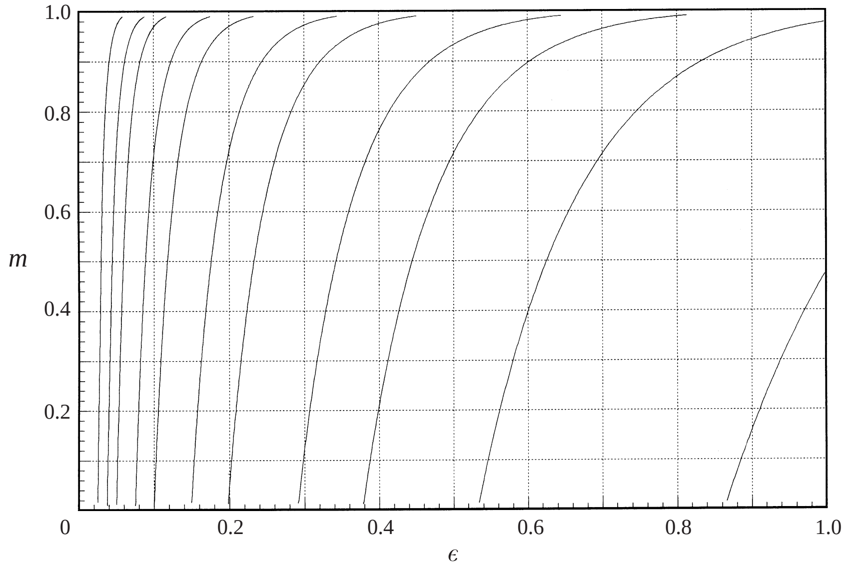

FIGURE 1. The condition (4.8). The parameter m versus the wave steepness ϵ. The curves are curves of constant n, starting from the right $n = 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48$.

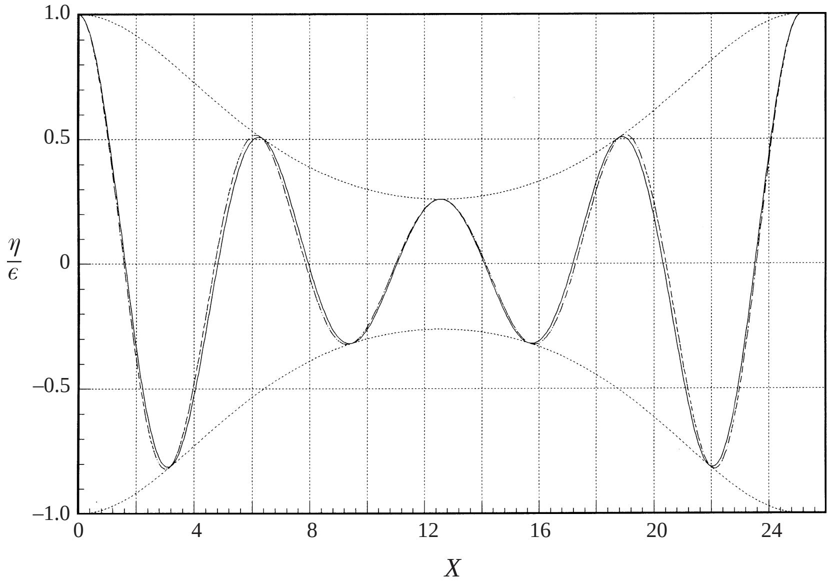

FIGURE 2. The surface elevation as a function of $X = \epsilon(x - c_g t)$, for $\epsilon = 0.5$ and $n = 4$ ($m \approx 0.932$). - - -, The envelope given by (3.4); –––, the surface elevation without the higher-order correction $f_0(X)$; ——, the surface elevation with the higher-order correction $f_0(X)$.

The surface elevation correct to lowest order in ϵ as given by (4.11) is plotted for $\epsilon = 0.5$ and $n = 4$ ($m \approx 0.932$) in figure 2, with and without the higher-order correction $f_0(X)$.

It is seen that the higher-order correction of the surface elevation $f_0(X)$ does not change the overall picture, except that it somewhat increases/decreases the length

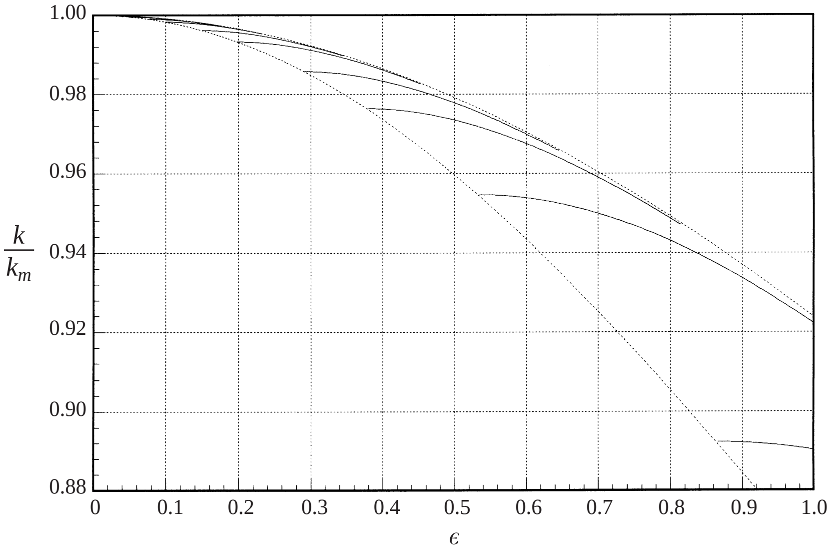

FIGURE 3. The wavenumber. ——, The normalized wavenumber k/k_m as function of the wave steepness ϵ: curves of constant n, starting from the bottom right corner $n = 1, 2, 3, 4, 6, 8, 12$. $\cdots\cdots$, The limiting curves for m: lower curve, $m = 0$, upper curve, $m = 1$.

of the higher/lower waves in the group, which is in agreement with the lengthening effect by nonlinearity.

4.5. *Determination of the wavenumber k and the corrected angular frequency Ω*

On the basis of figure 1 and the concluding remarks of § 4.1, we can determine curves of constant n for the physical values for the wavenumber k and the corrected angular frequency Ω. The wavenumber and the corrected angular frequency have been normalized with k_m and ω_m, respectively, and are illustrated in figures 3 and 4; k_m and ω_m are the physical values for the wavenumber and the angular frequency at the minimum phase speed ($\epsilon = 0$) and are given by

$$k_m = \left(\frac{g}{s}\right)^{1/2}, \tag{4.12a}$$

$$\omega_m = \left(\frac{4g^3}{s}\right)^{1/4}. \tag{4.12b}$$

Simple algebraic manipulations of the expressions for the wavenumber and the corrected angular frequency lead to

$$\frac{k}{k_m} = (4\alpha)^{-1/2} = \left(1 + \tfrac{11}{64}(2-m)\epsilon^2\right)^{-1/2}, \tag{4.13}$$

$$\frac{\Omega}{\omega_m} = (4\alpha)^{-3/4} = \left(1 + \tfrac{11}{64}(2-m)\epsilon^2\right)^{-3/4}. \tag{4.14}$$

As an upper limit for ϵ we use $\epsilon = 1$ as in figure 1 and once again the curves have been truncated near $m = 0$ and near $m = 1$ to indicate that they do not reach these values. Equations (4.13) and (4.14) for the values $m = 0, 1$ are indicated by dotted curves in the figures.

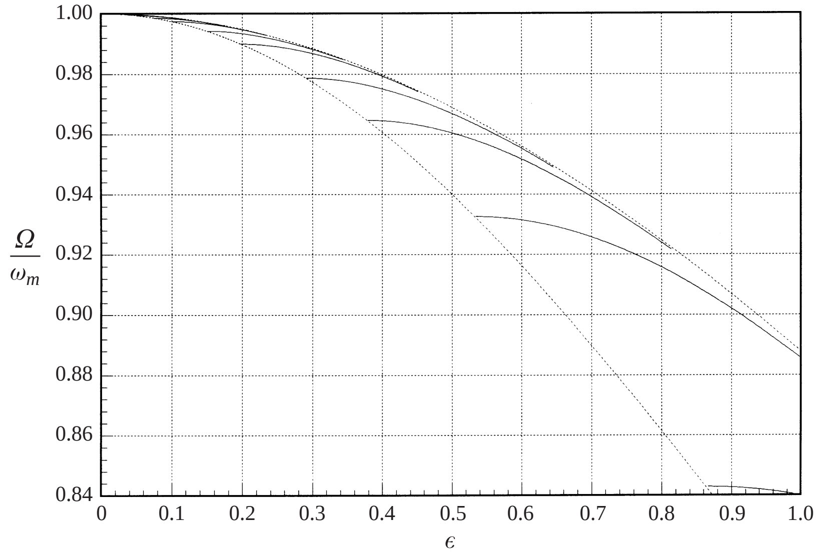

FIGURE 4. The corrected angular frequency. ——, The normalized corrected angular frequency Ω/ω_m as function of the wave steepness ϵ: curves of constant n, starting from the bottom right corner $n = 1, 2, 3, 4, 6, 8, 12$. $\cdots\cdots$, The limiting curves for m: lower curve, $m = 0$, upper curve, $m = 1$.

The curves of constant n start near the curve for $m = 0$ from where the curves for the wavenumber k and the corrected angular frequency Ω decrease when increasing ϵ, and they tend asymptotically towards the curve for $m = 1$.

Following our previous example with $\epsilon = 0.5$ and $n = 4$ ($m \approx 0.932$), we find that the normalized wavenumber is $k/k_m \approx 0.978$, and that the normalized corrected angular frequency is $\Omega/\omega_m \approx 0.967$.

4.6. *Numerical example for water waves*

For a water–air interface the coefficient of surface tension is taken as $\sigma = 7.4 \times 10^{-2}\,\mathrm{N\,m^{-1}}$, and with a fluid density of $\rho = 1 \times 10^3\,\mathrm{kg\,m^{-3}}$, the coefficient of surface tension divided by the fluid density takes the value $s = 7.4 \times 10^{-5}\,\mathrm{m^3\,s^{-2}}$. Finally, the gravitational acceleration is set to be $g = 9.81\ \mathrm{m\ s^{-2}}$. Thus, the wavenumber and the angular frequency at the minimum phase speed can be calculated as $k_m = 364.10\ \mathrm{m^{-1}}$ ($L_m = 1.73$ cm) and $\omega_m = 84.520\ \mathrm{s^{-1}}$ ($T_m = 0.0743$ s), respectively. The phase speed of this wave is found to be $c_m = 0.23213\,\mathrm{m\ s^{-1}}$.

Following our example ($\epsilon = 0.5$ and $n = 4$, $m \approx 0.932$), we find that the wavenumber and the corrected angular frequency are $k = 356\ \mathrm{m^{-1}}$ and $\Omega = 81.7\,\mathrm{s^{-1}}$. These correspond to a wavelength, corrected wave period, and a corrected phase speed/group velocity of the entire group of: $L = 0.0706$ m, $T = 0.308$ s and $c = c_g = 0.230\ \mathrm{m\ s^{-1}}$. Finally, the amplitude of the wave can be calculated to $a = 0.00140$ m.

5. Comparison to other results

The corrected angular frequency, Ω, from now on is assumed to have the general form

$$\Omega = \omega - \omega_c, \tag{5.1}$$

where ω is the angular frequency determined by the linear dispersion relation (2.1), and ω_c is a nonlinear correction to the angular frequency.

For our weakly nonlinear solution ($0 \leqslant \epsilon \ll 1$), we found that the correction to the angular frequency is given by

$$\omega_c = \frac{(2-m)}{2} q\epsilon^2 \tag{5.2}$$

(see (3.11)).

By combining (5.1) and the linear dispersion relation (2.1), and by assuming that the corrected phase speed ($c = \Omega/k$) has been normalized to 1, we obtain the equation

$$k^3 - k^2 + (\alpha - 2\omega_c)k - \omega_c^2 = 0, \tag{5.3}$$

which is a nonlinear dispersion relationship.

Our weakly nonlinear calculations in §§2, 3 and 4 are compatible with the following approximation of (5.3):

$$k^3 - k^2 + (\alpha - 2\omega_c)k = 0, \tag{5.4}$$

which has three roots:

$$k' = 0, \quad \left.\begin{matrix} k'' \\ k''' \end{matrix}\right\} = \frac{1 \pm (1 - 4(\alpha - 2\omega_c))^{1/2}}{2}. \tag{5.5}$$

In our solution $\alpha - 2\omega_c = \alpha_0 = \frac{1}{4}$ and $k'' = k''' = \frac{1}{2}$. This is also the case for the solution of Akylas *et al.* (1998).

Other authors have treated cases for which $\alpha - 2\omega_c \neq \alpha_0 = \frac{1}{4}$. The linearized version of their solution yields

$$\eta = a'' \cos\left(k''(x-ct)\right) + a''' \cos\left(k'''(x-ct)\right), \tag{5.6}$$

where k'' k''' are given by (5.5). When $a'' = a''' = a_0/2$ this solution simplifies to

$$\eta = a_0 \cos\left(k_0(x-ct)\right) \cos\left(\Delta k(x-ct)\right), \tag{5.7}$$

where $k_0 = \frac{1}{2}$ and $\Delta k = (1 - 4(\alpha - \omega_c))^{1/2}/2$. The above solution is seen to be a periodic solution with a carrier wavenumber of k_0 and with a modulation wavenumber of Δk. Furthermore it is seen that the carrier wave and the modulation wave travel at the same speed, as is the case for the periodic waves described in the previous sections. When $\alpha = 2/9$ and $\omega_c = 0$, $k'' = 1/3$ and $k''' = 2k'' = 2/3$ yield the well-known Wilton's ripple.

As Zufiria (1987) considers the Wilton's ripple ($\alpha = 2/9$) among other periodic waves, we find it most probable that it is members of this family of periodic wave forms on which his numerical study focuses. The periodic wave forms mentioned in Dias (1994) are also members of this family of periodic solutions. However, further investigation of this family of periodic waves falls outside the aims of this study.

6. Summary and concluding remarks

In the present note we have described periodic gravity–capillary wave forms propagating near the minimum phase speed. These wave forms are obtained as solutions to the governing Schrödinger-type equation, which was derived by Hogan (1985) and used in Akylas *et al.* (1998).

These periodic wave forms are described by the three parameters ϵ, m and n, which are the wave steepness, a parameter describing the form of the envelope and an integer number of waves per modulation wavelength. Only two of these three parameter can be chosen freely as shown in figure 1.

Knowing two out of these three parameters, the other wave characteristics like the wavenumber, the corrected angular frequency and the wave amplitude can be easily determined according to figures 3 and 4 and the expression $a = \epsilon/k$.

In the limits of the m-parameter (0 and 1) the periodic wave form simplifies into a Stokes' wave train and the solitary wave form described in e.g. Akylas *et al.* (1998), respectively. The phase correction in terms of the function f requires, however, special treatment in the limits of the m-parameter.

Finally we found that the periodic solutions described in the present note, and the periodic solutions described in e.g. the numerical study of Zufiria (1987) most probably are not of the same kind.

In 1992 Benjamin proposed an integro-differential equation for interfacial waves in a two-fluid system where the interface is subject to capillarity effects; in 1996 Benjamin proved that this equation exhibited periodic solutions, but did not present an explicit expression for these periodic solutions. As Akylas *et al.* (1998) reformulated the integro-differential equation of Benjamin (1992) into a Schrödinger-type equation similar to the one studied here, the periodic solution discussed in the present note may turn out to be related to those mentioned by Benjamin (1996).

In this study we restricted ourselves to n (n integer) waves per modulation wavelength, but extending the work to n (n integer) waves per N (N integer) modulation wavelengths is a straightforward task. We also restricted ourselves to symmetric wave forms, but the introduction of an arbitrary phase shift should be a straightforward task for the periodic wave forms described here ($0 < m < 1$).

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